

On the stability of thin-shell wormholes

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A thin-shell wormhole is theoretically constructible by surgically grafting together two Schwarzschild spacetimes using the so-called cut-and-paste technique. By describing such a wormhole as the limiting case of a spherical shell, it is shown that the structure must be unstable to linearized radial perturbations. Some earlier studies by the author et al. have shown, however, that under certain conditions, thin-shell wormholes can be stable.

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I. INTRODUCTION

It is well known that the thin-shell wormholes due to Visser [1] are constructed by the so-called cut-and-paste technique: start with two copies of Schwarzschild spacetime and remove from each manifold the four-dimensional regions

$$\Omega^\pm = \{r \leq a \mid a > 2M\}, \quad (1)$$

where a is a constant. We now identify (in the sense of topology) the timelike hypersurfaces

$$\partial\Omega^\pm = \{r = a \mid a > 2M\}. \quad (2)$$

The resulting manifold is geodesically complete and has two asymptotically flat regions connected by the throat $r = a$.

The subsequent stability analysis depends on the assumption that the radius of the throat is a function of time, so that the induced metric is

$$ds^2 = -d\tau^2 + [a(\tau)]^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (3)$$

where τ is the proper time on the junction surface. So it becomes convenient to denote $da/d\tau$ by \dot{a} .

The stability analysis is carried out by plotting the radius $r = a$ against β^2 , where β is usually interpreted as the speed of sound. There are regions of stability for $\beta^2 \geq 3/2 + \sqrt{5}$ and $\beta^2 \leq -1/2$. While β would normally be confined to the interval $(0, 1]$, Visser notes that we are dealing with exotic matter whose properties are not well understood. In fact, β^2 could be just a convenient parameter.

The purpose of this note is to study a wormhole restricted to a spherical shell that can be made arbitrarily thin and having the appropriate surface stresses, thereby approximating a thin shell due to Schwarzschild surgery, and to show that such a wormhole is indeed unstable to linearized radial perturbations. Connections to other studies of thin-shell wormholes are discussed at the end. (A more general discussion of the properties of thin-shell wormholes and their stability can be found in Refs. [2] and [3].)

II. WORMHOLE STRUCTURE

We start with the static and spherically symmetric line element

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (4)$$

(We are using geometrized units: $c = G = 1$.) Here $\Phi(r)$ is called the *redshift function*, which must be everywhere finite to prevent an event horizon. The function $b(r)$ is called the *shape function* because it determines the shape of the wormhole when viewed, for example, in an embedding diagram. The shape function must satisfy the following conditions: (1) $b(r_0) = r_0$, where $r = r_0$ is the *throat* of the wormhole, (2) the *flare-out* condition $b'(r_0) < 1$, and (3) $b(r) < r$ for $r > r_0$. The flare-out condition can only be satisfied by violating the null energy condition [4].

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The Einstein field equations are stated next:

$$\frac{b'}{r^2} = 8\pi\rho, \quad (5)$$

$$-\frac{b}{r^3} + 2\left(1 - \frac{b}{r}\right)\frac{\Phi'}{r} = 8\pi p_r, \quad (6)$$

and

$$\left(1 - \frac{b}{r}\right)\left[\Phi'' + (\Phi')^2 - \frac{b'r - b}{2r(r-b)}\Phi' - \frac{b'r - b}{2r^2(r-b)} + \frac{\Phi'}{r}\right] = 8\pi p_t. \quad (7)$$

For now we will assume that the density ρ is constant and that the wormhole material is confined to the spherical shell $r_0 \leq r \leq a$, i.e.,

$$\rho(r) = \begin{cases} \rho_0, & r_0 \leq r \leq a \\ 0, & r > a, \end{cases} \quad (8)$$

where ρ_0 is a constant and $r = r_0$ is the throat. (This form of $\rho(r)$ was also considered by Sushkov [5] in the discussion of wormholes supported by phantom energy.) Now from Eq. (5), we obtain the shape function

$$b(r) = \frac{8\pi r^3}{3}\rho_0 - \frac{8\pi r_0^3}{3} + r_0. \quad (9)$$

Observe that $b(r_0) = r_0$. Since we are using geometrized units, we can expect ρ_0 to be quite small compared to, say, r_0 . Hence

$$b'(r_0) = 8\pi r_0^2 \rho_0 < 1, \quad (10)$$

thereby satisfying the flare-out condition.

III. JUNCTION TO AN EXTERNAL VACUUM SOLUTION

Since the wormhole material is cut off at $r = a$, the interior solution must be matched to the exterior Schwarzschild solution

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - 2M/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (11)$$

Since the metric coefficients are continuous, we must have in view of Eq. (4),

$$M = \frac{1}{2}b(a) = \frac{4\pi a^3}{3}\rho_0 - \frac{4\pi r_0^3}{3}\rho_0 + \frac{1}{2}r_0 \quad (12)$$

and

$$\Phi(a) = \frac{1}{2}\ln\left(1 - \frac{2M}{a}\right). \quad (13)$$

(The components $g_{\theta\theta}$ and $g_{\phi\phi}$ are already continuous due to the spherical symmetry.)

At this point we need to turn our attention to surface stresses, since these will place some severe restrictions on the wormhole geometry. To see why, let us recall the Lanczos equations [6]

$$\sigma = -\frac{1}{4\pi}\kappa^\theta_\theta \quad (14)$$

and

$$\mathcal{P} = \frac{1}{8\pi}(\kappa^\tau_\tau + \kappa^\theta_\theta), \quad (15)$$

where $\kappa_{ij} = K_{ij}^+ - K_{ij}^-$ and K_{ij} is the extrinsic curvature. According to Ref. [6],

$$\kappa_\theta = \frac{1}{a} \sqrt{1 - \frac{2M}{a}} - \frac{1}{a} \sqrt{1 - \frac{b(a)}{a}}.$$

So by Eq. (14),

$$\sigma = -\frac{1}{4\pi a} \left(\sqrt{1 - \frac{2M}{a}} - \sqrt{1 - \frac{b(a)}{a}} \right). \quad (16)$$

In view of Eq. (12) one could reasonably expect that $\sigma = 0$. However, part of the junction formalism is to assume that the junction surface $r = a$ is an infinitely thin surface having a nonzero density that may be positive or negative. Hence its mass is given by

$$m_s = 4\pi a^2 \sigma = -a \left(\sqrt{1 - \frac{2M}{a}} - \sqrt{1 - \frac{b(a)}{a}} \right). \quad (17)$$

As a check, if $\sigma(a) < 0$ (resp. $\sigma(a) > 0$), then $M < \frac{1}{2}b(a)$ (resp. $M > \frac{1}{2}b(a)$), and m_s is negative (resp. positive).

Next,

$$K_\tau^{\tau+} = \frac{M/a^2}{\sqrt{1 - 2M/a}}$$

and

$$K_\tau^{\tau-} = \Phi'(a) \sqrt{1 - \frac{b(a)}{a}}.$$

So the surface pressure is given by

$$\begin{aligned} \mathcal{P} &= \frac{1}{8\pi} \left[\frac{M/a^2}{\sqrt{1 - 2M/a}} - \Phi'(a) \sqrt{1 - \frac{b(a)}{a}} + \frac{1}{a} \sqrt{1 - \frac{2M}{a}} - \frac{1}{a} \sqrt{1 - \frac{b(a)}{a}} \right] \\ &= \frac{1}{8\pi} \left[\frac{M/a^2 - \Phi'(a) \sqrt{1 - 2M/a} \sqrt{1 - b(a)/a}}{\sqrt{1 - 2M/a}} + \frac{1}{a} \sqrt{1 - \frac{2M}{a}} - \frac{1}{a} \sqrt{1 - \frac{b(a)}{a}} \right] \\ &\approx \frac{1}{8\pi} \frac{M/a^2 - \Phi'(a)(1 - 2M/a)}{\sqrt{1 - 2M/a}} \end{aligned} \quad (18)$$

since $b(a) \approx 2M$. Before substituting the expression for M from Eq. (12), recall that ρ_0 is relatively small, so that

$$\mathcal{P} \approx \frac{1}{8\pi \sqrt{1 - 2M/a}} \left(\frac{r_0/2}{a^2} - \Phi'(a) + \Phi'(a) \frac{r_0}{a} \right). \quad (19)$$

It is easily shown that $\mathcal{P} > 0$ if, and only if, $\sigma(a) < 0$. According to Eq. (19), \mathcal{P} remains positive even if a is arbitrarily close to r_0 , i.e.,

$$\lim_{a \rightarrow r_0} \mathcal{P} > 0.$$

It follows that $\sigma(a) < 0$.

To draw a more general conclusion, suppose we assume that a is indeed close to r_0 , thereby forming a thin spherical shell. On the small interval $[r_0, a]$, any function $\rho(r)$ is approximately constant, so that the assumption of a constant density can be omitted. Moreover, since the interval $[r_0, a]$ can be made arbitrarily small, the nonzero surface stresses result in a wormhole that can be said to approximate a thin-shell wormhole. (Recall that a junction surface is called a thin shell whenever the surface stresses are nonzero, otherwise a boundary surface.) This wormhole differs from the usual cut-and-paste thin-shell wormhole in the sense that there exists a complete description in the form of a line element:

$$ds^2 = \begin{cases} -e^{2\Phi(r)} dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2), & r_0 \leq r \leq a, \\ -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - 2M/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2), & r > a, \end{cases} \quad (20)$$

where $b(r)$ is given in Eq. (9) and M in Eq. (12), while $\Phi(a) = \frac{1}{2} \ln(1 - 2M/a)$.

IV. STABILITY ANALYSIS

It was noted earlier that the junction surface $r = a$ is a function of proper time τ . Accordingly, following Lobo and Crawford [7], the density takes on the form

$$\sigma = -\frac{1}{4\pi a} \left(\sqrt{1 - \frac{2M}{a} + \dot{a}^2} - \sqrt{1 - \frac{b(a)}{a} + \dot{a}^2} \right), \quad (21)$$

where $\dot{a} = da/d\tau$. A similar adjustment can be made for \mathcal{P} .

To obtain the stability criterion, one starts by rearranging Eq. (21),

$$\sqrt{1 - \frac{2M}{a} + \dot{a}^2} = \sqrt{1 - \frac{b(a)}{a} + \dot{a}^2} - 4\pi a \sigma,$$

in order to obtain the equation of motion

$$\dot{a}^2 + V(a) = 0, \quad (22)$$

where $V(a)$ is the potential. It can be shown that

$$V(a) = 1 - \frac{M + b(a)/2}{a} - \frac{m_s^2}{4a^2} - \frac{(M - b(a)/2)^2}{m_s^2}, \quad (23)$$

using the notation in Ref. [7]. The idea is to linearize around a static solution $a = a_0$:

$$V(a) = V(a_0) + V'(a_0)(a - a_0) + \frac{1}{2}V''(a_0)(a - a_0)^2 + \text{higher-order terms.}$$

According to Ref. [7], a long and tedious calculation shows that $V(a_0) = 0$ and $V'(a_0) = 0$. So the stability criterion is

$$V''(a_0) > 0. \quad (24)$$

The question now is how to make best use of Eq. (23). Substituting the expression for m_s from Eq. (17) at $a = a_0$ merely confirms that $V(a_0) = 0$. So we are going to assume that m_s is a constant and then take advantage of the rather simple form of the shape function.

A straightforward calculation shows that

$$V''(a_0) = -\frac{2M}{a_0^3} - \frac{r_0}{a_0^3} - \frac{1}{a_0^3} \left(\frac{8\pi a_0^3}{3} - \frac{8\pi r_0^3}{3} \right) \rho_0 - \frac{3m_s^2}{2a_0^4} + \frac{1}{m_s^2} \left[\left(M - \frac{1}{2}b(a_0) \right) b''(a_0) - \frac{1}{2}(b'(a_0))^2 \right]. \quad (25)$$

In the last term, observe that $|M - \frac{1}{2}b(a_0)|$ is typically much smaller than $(b'(a_0))^2$, making all the terms negative. We conclude that the wormhole is unstable to linearized radial perturbations, regardless of how one interprets the parameter β in Sec. I.

V. CONCLUSION

A thin-shell wormhole is theoretically constructible by the so-called cut-and-paste technique: surgically graft two Schwarzschild black holes together to form a structure that is geodesically complete. The type of wormhole discussed in this note can be described by the line element in Eq. (20). It consists of a spherical shell that can be made arbitrarily thin, and so, by continuity, can be viewed as an approximation of the cut-and-paste thin-shell wormhole. Both are unstable to linearized radial perturbations.

Because of the instability, it is important to note that an earlier paper by the author [8] has shown that a noncommutative-geometry background produces a small region of stability around $a = a_0$ for a thin-shell wormhole. Two earlier papers by Usmani et al. [9] and Rahaman et al. [10] have also addressed the stability issue. According to Ref. [9], a stable region exists for thin-shell wormholes constructed from black holes in generalized dilaton-axion gravity. Analogous results hold for regular charged black holes [10]. (The stability region is shown

correctly in Ref. [10] but not in the arXiv version.) So under certain conditions, thin-shell wormholes can be stable to linearized radial perturbations.

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